Power of hypothesis tests

Packages

```r
library(tidyverse)
```

Errors in testing

What can happen:

	Decision	
Truth	**Do not reject**	**Reject null**
Null true	Correct	Type I error
Null false	Type II error	Correct

Tension between truth and decision about truth (imperfect).

- Prob. of type I error denoted α. Usually fix α, eg. $\alpha = 0.05$.
- Prob. of type II error denoted β. Determined by the planned experiment. Low β good.
- Prob. of not making type II error called **power** ($= 1 - \beta$). *High* power good.

Power 1/2

- Suppose $H_0 : \theta = 10$, $H_a : \theta \neq 10$ for some parameter θ.
- Suppose H_0 wrong. What does that say about θ?
- Not much. Could have $\theta = 11$ or $\theta = 8$ or $\theta = 496$. In each case, H_0 wrong.

Power 2/2

- How likely a type II error is depends on what θ is:
  - If $\theta = 496$, should reject $H_0 : \theta = 10$ even for small sample, so β small (power large).
  - If $\theta = 11$, hard to reject H_0 even with large sample, so β would be larger (power smaller).
- Power depends on true parameter value, and on sample size.
- So we play "what if": "if θ were 11 (or 8 or 496), what would power be?".

Figuring out power

- Time to figure out power is before you collect any data, as part of planning process.
- Need to have idea of what kind of departure from null hypothesis of interest to you, eg. average improvement of 5 points on reading test scores. (Subject-matter decision, not statistical one.)
- Then, either:
  - "I have this big a sample and this big a departure I want to detect. What is my power for detecting it?"
  - "I want to detect this big a departure with this much power. How big a sample size do I need?"

How to understand/estimate power?

- Suppose we test $H_0 : \mu = 10$ against $H_a : \mu \neq 10$, where μ is population mean.
- Suppose in actual fact, $\mu = 8$, so H_0 is wrong. We want to reject it. How likely is that to happen?
- Need population SD (take $\sigma = 4$) and sample size (take $n = 15$). In practice, get σ from pilot/previous study, and take the n we plan to use.
- Idea: draw a random sample from the true distribution, test whether its mean is 10 or not.
- Repeat previous step "many" times: simulation.

Making it go

- Random sample of 15 normal observations with mean 8 and SD 4:

```
x <- rnorm(15, 8, 4)
x
```

```
 [1] 14.487469  5.014611  6.924277  5.201860  8.852952 10.8
 [8] 11.165242  8.016188 12.383518  1.378099  3.172503 13.0
[15]  5.015575
```

- Test whether `x` from population with mean 10 or not (over):

...continued

```
t.test(x, mu = 10)
```

```

	One Sample t-test

data:  x
t = -1.8767, df = 14, p-value = 0.08157
alternative hypothesis: true mean is not equal to 10
95 percent confidence interval:
  5.794735 10.280387
sample estimates:
mean of x 
 8.037561 
```

- P-value 0.081, so fail to reject the mean being 10 (a Type II error).

or get just P-value

```
ans <- t.test(x, mu = 10)
ans$p.value
```

```
[1] 0.0815652
```

Run this lots of times via simulation

- draw random samples from the truth
- test that $\mu = 10$
- get P-value
- Count up how many of the P-values are 0.05 or less.

In code

```
tibble(sim = 1:1000) %>%
  rowwise() %>%
  mutate(my_sample = list(rnorm(15, 8, 4))) %>%
  mutate(t_test = list(t.test(my_sample, mu = 10))) %>%
  mutate(p_val = t_test$p.value) %>%
  count(p_val <= 0.05)
```

```
# A tibble: 2 x 2
# Rowwise:
  `p_val <= 0.05`     n
  <lgl>           <int>
1 FALSE             578
2 TRUE              422
```

We correctly rejected 422 times out of 1000, so the estimated power is 0.422.

Try again with bigger sample

```
tibble(sim = 1:1000) %>%
  rowwise() %>%
  mutate(my_sample = list(rnorm(40, 8, 4))) %>%
  mutate(t_test = list(t.test(my_sample, mu = 10))) %>%
  mutate(p_val = t_test$p.value) %>%
  count(p_val <= 0.05)
```

```
# A tibble: 2 x 2
# Rowwise:
  `p_val <= 0.05`     n
  <lgl>           <int>
1 FALSE             119
2 TRUE              881
```

Power is (much) larger with a bigger sample.

How accurate is my simulation?

- At our chosen α, each simulated test independently either rejects or not with some probability p that I am trying to estimate (the power)
- Estimating a population probability using the sample proportion (the number of simulated rejections out of the number of simulated tests)
- hence, `prop.test`.
- inputs: number of rejections, number of simulations.

Sample size 15, rejected 422 times

```
prop.test(422, 1000)
```

```

	1-sample proportions test with continuity correction

data:  422 out of 1000, null probability 0.5
X-squared = 24.025, df = 1, p-value = 9.509e-07
alternative hypothesis: true p is not equal to 0.5
95 percent confidence interval:
 0.3912521 0.4533546
sample estimates:
    p
0.422
```

95% CI for power: 0.391 to 0.453

To estimate power more accurately

- Run more *simulations*:

Change 1000 to eg 10,000:

```
tibble(sim = 1:10000) %>%
  rowwise() %>%
  mutate(my_sample = list(rnorm(15, 8, 4))) %>%
  mutate(t_test = list(t.test(my_sample, mu = 10))) %>%
  mutate(p_val = t_test$p.value) %>%
  count(p_val <= 0.05)
```

```
# A tibble: 2 x 2
# Rowwise:
  `p_val <= 0.05`     n
  <lgl>           <int>
1 FALSE            5647
2 TRUE             4353
```

Accuracy of power now

```
prop.test(4353, 10000)
```

```

    1-sample proportions test with continuity correction

data:  4353 out of 10000, null probability 0.5
X-squared = 167.18, df = 1, p-value < 2.2e-16
alternative hypothesis: true p is not equal to 0.5
95 percent confidence interval:
 0.4255594 0.4450905
sample estimates:
     p
0.4353
```

0.426 to 0.445, about factor $\sqrt{10}$ shorter because number of simulations 10 times bigger.

Calculating power

- Simulation approach very flexible: will work for any test. But answer different each time because of randomness.
- In some cases, for example 1-sample and 2-sample t-tests, power can be calculated.
- `power.t.test`. Input `delta` is difference between null and true mean:

```
power.t.test(n = 15, delta = 10-8, sd = 4, type = "one.samp
```

```

     One-sample t test power calculation

              n = 15
          delta = 2
             sd = 4
      sig.level = 0.05
          power = 0.4378466
    alternative = two.sided
```

Comparison of results

Method	Power
Simulation (10000)	0.4353
`power.t.test`	0.4378

- Simulation power is similar to calculated power; to get more accurate value, repeat more times (eg. 100,000 instead of 10,000), which takes longer.
- With this small a sample size, the power is not great. With a bigger sample, the sample mean should be closer to 8 most of the time, so would reject $H_0 : \mu = 10$ more often.

Calculating required sample size

- Often, when planning a study, we do not have a particular sample size in mind. Rather, we want to know how big a sample to take. This can be done by asking how big a sample is needed to achieve a certain power.
- The simulation approach does not work naturally with this, since you have to supply a sample size.
  - For that, you try different sample sizes until you get power close to what you want.
- For the power-calculation method, you supply a value for the power, but leave the sample size missing.
- Re-use the same problem: $H_0 : \mu = 10$ against 2-sided alternative, true $\mu = 8$, $\sigma = 4$, but now aim for power 0.80.

Using power.t.test

- No `n=`, replaced by a `power=`:

```
power.t.test(power=0.80, delta=10-8, sd=4, type="one.sample
```

```

     One-sample t test power calculation

              n = 33.3672
          delta = 2
             sd = 4
      sig.level = 0.05
          power = 0.8
    alternative = two.sided
```

- Sample size must be a whole number, so round up to 34 (to get at least as much power as you want).

Power curves

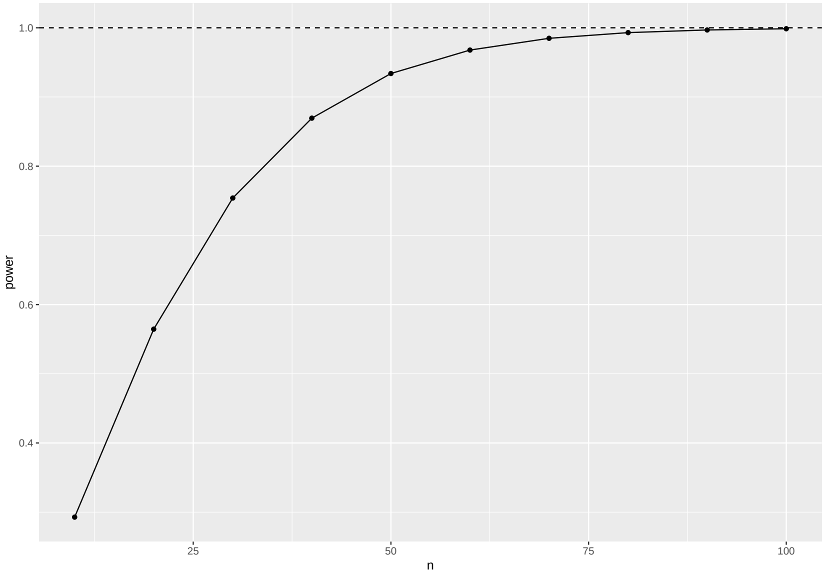
- Rather than calculating power for one sample size, or sample size for one power, might want a picture of relationship between sample size and power.
- Or, likewise, picture of relationship between difference between true and null-hypothesis means and power.
- Called power curve.
- Build and plot it yourself.

Building it:

```
tibble(n=seq(10, 100, 10)) %>% rowwise() %>%
  mutate(power_output =
           list(power.t.test(n = n, delta = 10-8, sd = 4,
                             type = "one.sample"))) %>%
  mutate(power = power_output$power) %>%
  ggplot(aes(x=n, y=power)) + geom_point() + geom_line() +
    geom_hline(yintercept=1, linetype="dashed") -> g2
```

The power curve

```
g2
```



Power curves for means

- Can also investigate power as it depends on what the true mean is (the farther from null mean 10, the higher the power will be).
- Investigate for two different sample sizes, 15 and 30.
- First make all combos of mean and sample size:

```
means <- seq(6,10,0.5)
ns <- c(15,30)
combos <- crossing(mean=means, n=ns)
```

The combos

```
combos
```

```
# A tibble: 18 x 2
    mean     n
   <dbl> <dbl>
 1   6      15
 2   6      30
 3   6.5    15
 4   6.5    30
 5   7      15
 6   7      30
 7   7.5    15
 8   7.5    30
 9   8      15
10   8      30
11   8.5    15
12   8.5    30
13   9      15
14   9      30
15   9.5    15
16   9.5    30
17  10      15
18  10      30
```

Calculate powers:

```
combos %>%
  rowwise() %>%
  mutate(power_stuff = list(power.t.test(n=n, delta=10-mean
                                         type="one.sample"))) %>%
  mutate(power = power_stuff$power) -> powers
```

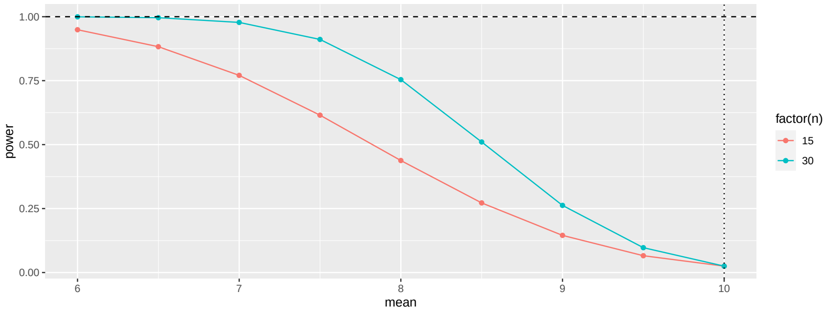
then make the plot:

```
g  <-  ggplot(powers, aes(x = mean, y = power, colour = fac
  geom_point() + geom_line() +
  geom_hline(yintercept = 1, linetype = "dashed") +
  geom_vline(xintercept = 10, linetype = "dotted")
```

- Need `n` as categorical so that `colour` works properly.

The power curves

```
g
```



Comments

- When `mean=10`, that is, the true mean equals the null mean, H_0 is actually true, and the probability of rejecting it then is $\alpha = 0.05$.
- As the null gets more wrong (mean decreases), it becomes easier to correctly reject it.
- The blue power curve is above the red one for any mean < 10, meaning that no matter how wrong H_0 is, you always have a greater chance of correctly rejecting it with a larger sample size.
- Previously, we had $H_0 : \mu = 10$ and a true $\mu = 8$, so a mean of 8 produces power 0.42 and 0.80 as shown on the graph.
- With $n = 30$, a true mean that is less than about 7 is almost certain to be correctly rejected. (With $n = 15$, the true mean needs to be less than 6.)

Two-sample power

- For kids learning to read, had sample sizes of 22 (approx) in each group
- and these group SDs:

```
kids %>% group_by(group) %>%
  summarize(n=n(), s=sd(score))
```

```
# A tibble: 2 x 3
  group     n     s
  <chr> <int> <dbl>
1 c        23  17.1
2 t        21  11.0
```

Setting up

- suppose a 5-point improvement in reading score was considered important (on this scale)
- in a 2-sample test, null (difference of) mean is zero, so `delta` is true difference in means
- what is power for these sample sizes, and what sample size would be needed to get power up to 0.80?
- SD in both groups has to be same in `power.t.test`, so take as 14.

Calculating power for sample size 22 (per group)

```
power.t.test(n=22, delta=5, sd=14, type="two.sample",
             alternative="one.sided")
```

```
     Two-sample t test power calculation

              n = 22
          delta = 5
             sd = 14
      sig.level = 0.05
          power = 0.3158199
    alternative = one.sided

NOTE: n is number in *each* group
```

sample size for power 0.8

```r
power.t.test(power=0.80, delta=5, sd=14, type="two.sample",
             alternative="one.sided")
```

```
     Two-sample t test power calculation

              n = 97.62598
          delta = 5
             sd = 14
      sig.level = 0.05
          power = 0.8
    alternative = one.sided

NOTE: n is number in *each* group
```

Comments

- The power for the sample sizes we have is very small (to detect a 5-point increase).
- To get power 0.80, we need 98 kids in *each* group!